



UNIVERSITÀ  
DEGLI STUDI  
FIRENZE

# FLORE

## Repository istituzionale dell'Università degli Studi di Firenze

### **Fourth-Order Differential Equation with Deviating Argument**

Questa è la Versione finale referata (Post print/Accepted manuscript) della seguente pubblicazione:

*Original Citation:*

Fourth-Order Differential Equation with Deviating Argument / M. Bartusek; M. Cecchi; Z. Dosla; M. Marini. - In: ABSTRACT AND APPLIED ANALYSIS. - ISSN 1085-3375. - ELETTRONICO. - 2012, Article ID 185242:(2012), pp. 1-17. [10.1155/2012/185242]

*Availability:*

This version is available at: 2158/891731 since:

*Published version:*

DOI: 10.1155/2012/185242

*Terms of use:*

Open Access

La pubblicazione è resa disponibile sotto le norme e i termini della licenza di deposito, secondo quanto stabilito dalla Policy per l'accesso aperto dell'Università degli Studi di Firenze (<https://www.sba.unifi.it/upload/policy-oa-2016-1.pdf>)

*Publisher copyright claim:*

(Article begins on next page)

## Research Article

# Fourth-Order Differential Equation with Deviating Argument

**M. Bartušek,<sup>1</sup> M. Cecchi,<sup>2</sup> Z. Došlá,<sup>1</sup> and M. Marini<sup>2</sup>**

<sup>1</sup> Department of Mathematics and Statistics, Masaryk University, 61137 Brno, Czech Republic

<sup>2</sup> Department of Electronics and Telecommunications, University of Florence, 50139 Florence, Italy

Correspondence should be addressed to Z. Došlá, dosla@math.muni.cz

Received 6 December 2011; Accepted 13 January 2012

Academic Editor: Paul Eloe

Copyright © 2012 M. Bartušek et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the fourth-order differential equation with middle-term and deviating argument  $x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f(x(\varphi(t))) = 0$ , in case when the corresponding second-order equation  $h'' + q(t)h = 0$  is oscillatory. Necessary and sufficient conditions for the existence of bounded and unbounded asymptotically linear solutions are given. The roles of the deviating argument and the nonlinearity are explained, too.

## 1. Introduction

The aim of this paper is to investigate the fourth-order nonlinear differential equation with middle-term and deviating argument

$$x^{(4)}(t) + q(t)x^{(2)}(t) + r(t)f(x(\varphi(t))) = 0. \quad (1.1)$$

The following assumptions will be made.

- (i)  $q$  is a continuously differentiable bounded away from zero function, that is,  $q(t) \geq q_0 > 0$  for large  $t$  such that

$$\int_0^\infty |q'(t)| dt < \infty. \quad (1.2)$$

- (ii)  $r, \varphi$  are continuous functions for  $t \geq 0$ ,  $r$  is not identically zero for large  $t$ ,  $\varphi(t) \geq 0$ , and  $\varphi(0) = 0$ ,  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ .

(iii)  $f$  is a continuous function such that  $f(u)u > 0$  for  $u \neq 0$ .

Observe that (i) implies that there exists a positive constant  $Q$  such that  $q(t) \leq Q$  and the linear second-order equation

$$h''(t) + q(t)h(t) = 0 \quad (1.3)$$

is oscillatory. Moreover, solutions of (1.3) are bounded together with their derivatives, see for example, [1, Theorem 2].

By a solution of (1.1) we mean a function  $x$  defined on  $[T_x, \infty)$ ,  $T_x \geq 0$ , which is differentiable up to the fourth order and satisfies (1.1) on  $[T_x, \infty)$  and  $\sup\{|x(t)| : t \geq T\} > 0$  for  $T \geq T_x$ .

A solution  $x$  of (1.1) is said to be *asymptotically linear (AL-solution)* if either

$$\lim_{t \rightarrow \infty} x(t) = c_x \neq 0, \quad \lim_{t \rightarrow \infty} x'(t) = 0, \quad (1.4)$$

or

$$\lim_{t \rightarrow \infty} |x(t)| = \infty, \quad \lim_{t \rightarrow \infty} x'(t) = d_x \neq 0, \quad (1.5)$$

for some constants  $c_x, d_x$ .

Fourth-order nonlinear differential equations naturally appear in models concerning physical, biological, and chemical phenomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement, see, for example, [2, 3].

When (1.3) is nonoscillatory and  $h$  is its eventually positive solution, it is known that (1.1) can be written as the two-term equation

$$\left( h^2(t) \left( \frac{x''(t)}{h(t)} \right)' \right)' + h(t)r(t)f(x(t)) = 0. \quad (1.6)$$

In this case, the question of oscillation and asymptotics of such class of equations has been investigated with sufficient thoroughness, see, for example, the papers [3–10] or the monographs [11, 12] and references therein.

Nevertheless, as far we known, there are only few results concerning (1.1) when (1.3) is oscillatory. For instance, the equation without deviating argument

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)f(x(t)) = 0 \quad (1.7)$$

has been investigated by Kiguradze in [13] in case  $q(t) \equiv 1$  and by the authors in [14, 15] when  $q$  satisfies (i). In particular, in [14] the oscillation of (1.1) in the case  $n = 3$  is studied. In [15], the existence of positive bounded and unbounded solutions as well as of oscillatory solutions for (1.7) has been considered and the case  $n = 4$  has been analyzed in detail. Other results can be found in [16] and references therein, in which the existence and uniqueness of almost periodic solutions for equations of type (1.1) with almost periodic coefficients  $q, r$  are studied.

Motivated by [14, 15], here we study the existence of AL-solutions for (1.1). The approach is completely different from the one used in [15], in which an iteration process, jointly with a comparison with the linear equation  $y^{(4)} + q(t)y^{(2)} = 0$ , is employed. Our tools are based on a topological method, certain integral inequalities, and some auxiliary functions. In particular, for proving the continuity in the Fréchet space  $C[t_0, \infty)$  of the fixed point operators here considered, we use a similar argument to that in the Vitali convergence theorem.

Our results extend to the case with deviating argument analogues ones stated in [15] for (1.7) when  $n = 4$ . We obtain sharper conditions for the existence of unbounded AL-solutions of (1.1), and, in addition, we show that under additional assumptions on  $q, r$ , these conditions become also necessary for the existence of AL-solutions, in both the bounded and unbounded cases. In the final part, we consider the particular case

$$f(u) = |u|^\lambda \operatorname{sgn} u \quad (\lambda > 0) \quad (1.8)$$

and we study the possible coexistence of bounded and unbounded AL-solutions. The role of deviating argument and the one of the growth of the nonlinearity are also discussed and illustrated by some examples.

## 2. Unbounded Solutions

Here we study the existence of unbounded AL-solutions of (1.1). Our first main result is the following.

**Theorem 2.1.** *For any  $c, 0 < c < \infty$ , there exists an unbounded solution  $x$  of (1.1) such that*

$$\lim_{t \rightarrow \infty} x'(t) = c, \quad \lim_{t \rightarrow \infty} x^{(i)}(t) = 0, \quad i = 2, 3, \quad (2.1)$$

*provided*

$$\int_0^\infty |r(t)|F(\varphi(t))dt < \infty, \quad (2.2)$$

*where for  $u > 0$*

$$F(u) = \max \left\{ f(v) : |v - u| \leq \frac{1}{2}u \right\}. \quad (2.3)$$

*Proof.* Without loss of generality, we prove the existence of solutions of (1.1) satisfying (2.1) for  $c = 1$ .

Let  $u$  and  $v$  be two linearly independent solutions of (1.3) with Wronskian  $d = 1$ . Denote

$$w(s, t) = u(s)v(t) - u(t)v(s), \quad z(s, t) = \frac{\partial}{\partial t} w(s, t). \quad (2.4)$$

As claimed by the assumptions on  $q$ , all solutions of (1.3) and their derivatives are bounded. Thus, put

$$M = \sup\{|w(s, t)| + |z(s, t)| : s \geq 0, t \geq 0\}, \quad L = \frac{2(2M + 1)}{q_0}. \quad (2.5)$$

Let  $\bar{t} \geq t_0$  be such that  $\varphi(t) \geq t_0$  for  $t \geq \bar{t}$ . Define

$$\bar{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \geq \bar{t}, \\ \varphi(\bar{t}) & \text{if } t_0 \leq t \leq \bar{t}, \end{cases} \quad (2.6)$$

and choose  $t_0 \geq 0$  large so that

$$\int_{t_0}^{\infty} |r(s)| F(\bar{\varphi}(s)) ds \leq \frac{1}{2L}, \quad \frac{1}{q_0} \int_{t_0}^{\infty} |q'(t)| dt \leq \frac{1}{2}. \quad (2.7)$$

Denote by  $C[t_0, \infty)$  the Fréchet space of all continuous functions on  $[t_0, \infty)$ , endowed with the topology of uniform convergence on compact subintervals of  $[t_0, \infty)$ , and consider the set  $\Omega \subset C[t_0, \infty)$  given by

$$\Omega = \left\{ x \in C[t_0, \infty) : \frac{t}{2} \leq x(t) \leq \frac{3t}{2} \right\}. \quad (2.8)$$

Let  $T > t_0$  and define on  $[t_0, T]$  the function

$$g(t) = \gamma''(t) + q(t)\gamma(t), \quad (2.9)$$

where

$$\gamma(t) = - \int_t^T \int_{\tau}^{\infty} r(s) f(x(\bar{\varphi}(s))) w(s, \tau) ds d\tau \quad (2.10)$$

and  $x \in \Omega$ . Then,

$$\gamma'(t) = \int_t^{\infty} r(s) f(x(\bar{\varphi}(s))) w(s, t) ds, \quad (2.11)$$

$$\begin{aligned} \gamma''(t) &= \int_t^{\infty} r(s) f(x(\bar{\varphi}(s))) z(s, t) ds, \\ \gamma'''(t) &= -r(t) f(x(\bar{\varphi}(t))) - q(t)\gamma'(t). \end{aligned} \quad (2.12)$$

Moreover,  $g(T) = \gamma''(T)$ , and it holds for  $t \in [t_0, T]$  that

$$g'(t) = \gamma'''(t) + q(t)\gamma'(t) + q'(t)\gamma(t) = -r(t) f(x(\bar{\varphi}(t))) + q'(t)\gamma(t). \quad (2.13)$$

Integrating, we obtain

$$g(t) = g(T) - \int_t^T g'(s)ds = \gamma''(T) + \int_t^T r(s)f(x(\bar{\varphi}(s)))ds - \int_t^T q'(s)\gamma(s)ds. \quad (2.14)$$

From here and (1.3), we get

$$\gamma(t) = \frac{1}{q(t)} \left( \gamma''(T) - \gamma''(t) + \int_t^T r(s)f(x(\bar{\varphi}(s)))ds - \int_t^T q'(s)\gamma(s)ds \right). \quad (2.15)$$

Thus,

$$|\gamma(t)| \leq \frac{1+2M}{q_0} \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds + \frac{1}{q_0} \max_{t \leq s \leq T} |\gamma(s)| \int_{t_0}^\infty |q'(s)|ds, \quad (2.16)$$

and so

$$\left( 1 - \frac{1}{q_0} \int_{t_0}^\infty |q'(s)|ds \right) \max_{t \leq s \leq T} |\gamma(s)| \leq \frac{1+2M}{q_0} \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds, \quad (2.17)$$

or, in view of (2.7),

$$|\gamma(t)| \leq L \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds. \quad (2.18)$$

Thus, from (2.10), as  $T \rightarrow \infty$ , we get

$$\left| \int_t^\infty \int_\tau^\infty r(s)f(x(\bar{\varphi}(s)))w(s,\tau)ds d\tau \right| \leq L \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds. \quad (2.19)$$

Hence, the operator  $\mathcal{T} : \Omega \rightarrow \Omega$  given by

$$\mathcal{T}(x)(t) = t - \int_{t_0}^t \int_\sigma^\infty \int_\tau^\infty r(s)f(x(\bar{\varphi}(s)))w(s,\tau)ds d\tau d\sigma \quad (2.20)$$

is well defined for any  $x \in \Omega$ . Moreover, in view of (2.19), we have

$$|\mathcal{T}'(x)(t) - 1| \leq L \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds. \quad (2.21)$$

From here, in virtue of (2.7) we get

$$|\mathcal{T}(x)(t) - t| \leq Lt \int_{t_0}^\infty |r(s)|F(\bar{\varphi}(s))ds \leq \frac{1}{2}t. \quad (2.22)$$

Hence,  $\mathcal{T}(\Omega) \subset \Omega$ . From (2.5) and (2.11), we have

$$|\mathcal{T}''(x)(t)| = |\gamma'(t)| \leq M \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds, \quad (2.23)$$

and so  $\lim_{t \rightarrow \infty} \mathcal{T}''(x)(t) = 0$ . Similarly,

$$|\mathcal{T}'''(x)(t)| = |\gamma''(t)| \leq M \int_t^\infty |r(s)|F(\bar{\varphi}(s))ds, \quad (2.24)$$

and thus,  $\lim_{t \rightarrow \infty} \mathcal{T}'''(x)(t) = 0$ , too. In addition,

$$\mathcal{T}^{(4)}(x)(t) = \gamma'''(t) = -q(t)\mathcal{T}''(x)(t) - r(t)f(x(\bar{\varphi}(t))). \quad (2.25)$$

Hence, any fixed point of  $\mathcal{T}$  is a solution of (1.1) for large  $t$ .

Let us show that  $\mathcal{T}(\Omega)$  is relatively compact, that is,  $\mathcal{T}(\Omega)$  consists of functions equibounded and equicontinuous on every compact interval of  $[t_0, \infty)$ . Because  $\mathcal{T}(\Omega) \subset \Omega$ , the equiboundedness follows. Moreover, in view of (2.7),  $\mathcal{T}'(u)(t)$  is bounded for any  $u \in \Omega$ , which yields the equicontinuity of the elements in  $\mathcal{T}(\Omega)$ .

Now we prove the continuity of  $\mathcal{T}$  in  $\Omega$ . Let  $\{x_n\}$ ,  $n \in \mathbb{N}$ , be a sequence in  $\Omega$ , which uniformly converges to  $\bar{x} \in \Omega$  on every compact interval of  $[t_0, \infty)$ . Fixing  $T > t_0$ , in virtue of (2.23), the dominated convergence Lebesgue theorem gives

$$\lim_{n \rightarrow \infty} \int_\sigma^T \int_\tau^\infty r(s)(f(x_n(\bar{\varphi}(s))))w(s, \tau)ds d\tau = \int_\sigma^T \int_\tau^\infty r(s)(f(\bar{x}(\bar{\varphi}(s))))w(s, \tau)ds d\tau. \quad (2.26)$$

Moreover,

$$\begin{aligned} & \left| \int_\sigma^\infty \int_\tau^\infty r(s)(f(x_n(\bar{\varphi}(s))))w(s, \tau)ds d\tau \right| \\ & \leq \left| \int_\sigma^T \int_\tau^\infty r(s)(f(x_n(\bar{\varphi}(s))))w(s, \tau)ds d\tau \right| \\ & \quad + \int_T^\infty \int_\tau^\infty |r(s)|(f(x_n(\bar{\varphi}(s)))) + f(\bar{x}(\bar{\varphi}(s))))|w(s, \tau)|ds d\tau. \end{aligned} \quad (2.27)$$

In view of (2.19), we have

$$\int_T^\infty \int_\tau^\infty |r(s)|(f(x_n(\bar{\varphi}(s)))) + f(\bar{x}(\bar{\varphi}(s))))|w(s, \tau)|ds d\tau \leq 2M \int_T^\infty |r(s)|F(\bar{\varphi}(s))ds. \quad (2.28)$$

Thus, choosing  $T$  sufficiently large, we get from (2.27)

$$\lim_{n \rightarrow \infty} \int_\sigma^\infty \int_\tau^\infty r(s)f(x_n(\bar{\varphi}(s))))w(s, \tau)ds d\tau = \int_\sigma^\infty \int_\tau^\infty r(s)f(\bar{x}(\bar{\varphi}(s))))w(s, \tau)ds d\tau, \quad (2.29)$$

and so the continuity of  $\mathcal{T}$  in  $\Omega$  follows. By the Tychonoff fixed point theorem, the operator  $\mathcal{T}$  has a fixed point  $x$ , which is an unbounded solution of (1.1) satisfying (2.1).  $\square$

*Remark 2.2.* With minor modifications, Theorem 2.1 gives also the existence of eventually negative unbounded AL-solutions. The details are omitted.

*Remark 2.3.* When  $\varphi(t) \equiv t$ , Theorem 2.1 is related with Theorem 1 in [15], from which the existence of unbounded AL-solutions of (1.1) can be obtained under stronger assumptions. A comparison between Theorem 1 in [15] and Theorem 2.1 is given in Section 4.

Our next result gives a necessary condition for the existence of unbounded solutions  $x$  of (1.1) satisfying for large  $t$  and some  $\alpha$  and  $\beta$

$$0 < \alpha \leq x'(t) \leq \beta. \quad (2.30)$$

**Theorem 2.4.** Assume either  $r(t) \geq 0$  or  $r(t) \leq 0$ .

Equation (1.1) does not have eventually positive solutions  $x$  satisfying (2.30) for large  $t$  and some  $\alpha$  and  $\beta$  provided

$$\int_0^\infty |r(t)| \bar{F}(\varphi(t)) dt = \infty, \quad (2.31)$$

where for  $u > 0$

$$\bar{F}(u) = \min \left\{ f(v) : \frac{\alpha}{2}u \leq v \leq 2\beta u \right\}. \quad (2.32)$$

*Proof.* Assume  $r(t) \geq 0$ , and let  $x$  be an eventually positive solution of (1.1) satisfying (2.30). Then, there exists  $\tau$  such that

$$\frac{\alpha}{2}t \leq x(t) \leq 2t\beta \quad \text{for } t \geq \tau. \quad (2.33)$$

Consequently, in view of (2.31), we have

$$\lim_{t \rightarrow \infty} \int_\tau^t r(s) f(x(\varphi(s))) ds = \infty. \quad (2.34)$$

Thus, integrating (1.1), we get

$$\lim_{t \rightarrow \infty} \left( x'''(t) + \int_\tau^t q(s) x''(s) ds \right) = -\infty. \quad (2.35)$$



Furthermore,

$$\begin{aligned} \left| \int_{\tau}^t q(s)x''(s)ds \right| &= \left| q(t)x'(t) - q(\tau)x'(\tau) - \int_{\tau}^t q'(s)x'(s)ds \right| \\ &\leq 2\beta Q + \beta \int_{\tau}^{\infty} |q'(s)|ds < \infty, \end{aligned} \quad (2.36)$$

where  $Q = \sup_{s \geq 0} q(s)$ . Hence  $\lim_{t \rightarrow \infty} x'''(t) = -\infty$ , which gives a contradiction with the boundedness of  $x'$ . Finally, if  $r(t) \leq 0$ , the argument is similar and the details are left to the reader.  $\square$

### 3. Bounded Solutions

In this section we study the existence of bounded AL-solutions of (1.1). The following holds.

**Theorem 3.1.** *If*

$$\int_0^{\infty} |r(t)|t dt < \infty, \quad (3.1)$$

*then, for any  $c \in \mathbb{R} \setminus \{0\}$ , there exists a solution  $x$  of (1.1) satisfying*

$$\lim_{t \rightarrow \infty} x(t) = c, \quad \lim_{t \rightarrow \infty} x^{(i)}(t) = 0, \quad i = 1, 2. \quad (3.2)$$

*Proof.* Without loss of generality, we prove the existence of solutions of (1.1) satisfying (3.2) for  $c = 1$ .

We proceed by a similar way to that in the proof of Theorem 2.1, and we sketch the proof.

Let  $M$  be the constant given in (2.5), and let

$$K = \max \left\{ f(u) : \frac{1}{2} \leq u \leq \frac{3}{2} \right\}, \quad L_1 = \frac{2K(2M+1)}{q_0}. \quad (3.3)$$

Choose  $t_0 \geq 0$  large so that

$$\int_{t_0}^{\infty} t|r(t)|dt \leq \frac{1}{2L_1}, \quad \frac{1}{q_0} \int_{t_0}^{\infty} |q'(s)|ds \leq \frac{1}{2}, \quad (3.4)$$

and define  $\bar{\varphi}$  as in (2.6). Denote by  $C[t_0, \infty)$  the Fréchet space of all continuous functions on  $[t_0, \infty)$ , endowed with the topology of uniform convergence on compact subintervals of  $[t_0, \infty)$ , and consider the set  $\Omega \subset C[t_0, \infty)$  given by

$$\Omega = \left\{ x \in C[t_0, \infty) : \frac{1}{2} \leq x(t) \leq \frac{3}{2} \right\}. \quad (3.5)$$

Let  $T > t_0$ , and, for any  $x \in \Omega$ , consider again the function  $\gamma$  given in (2.10). Reasoning as in the proof of Theorem 2.1, with minor changes, we obtain

$$\left| \int_t^\infty \int_\tau^\infty r(s) f(x(\bar{\varphi}(s))) w(s, \tau) ds d\tau \right| \leq L_1 \int_t^\infty |r(s)| ds. \quad (3.6)$$

Hence, in virtue of (3.1), the operator  $\mathcal{H} : \Omega \rightarrow \Omega$  given by

$$\mathcal{H}(x)(t) = 1 + \int_t^\infty \int_\sigma^\infty \int_\tau^\infty r(s) f(x(\bar{\varphi}(s))) w(s, \tau) ds d\tau d\sigma \quad (3.7)$$

is well defined and  $\lim_{t \rightarrow \infty} \mathcal{H}(x)(t) = 1$ . In view of (3.6), we get

$$|\mathcal{H}'(x)(t)| \leq L_1 \int_t^\infty |r(s)| ds. \quad (3.8)$$

A similar estimation holds for  $|\mathcal{H}''(x)|$ . Thus,  $\lim_{t \rightarrow \infty} \mathcal{H}^{(i)}(x)(t) = 0$ ,  $i = 1, 2$ . In view of (3.4), from (3.8), we obtain

$$|\mathcal{H}(x)(t) - 1| \leq L_1 \int_t^\infty s |r(s)| ds \leq \frac{1}{2}, \quad (3.9)$$

that is,  $\mathcal{H}(\Omega) \subset \Omega$ . Moreover, a standard calculation gives

$$\mathcal{H}^{(4)}(x)(t) = -q(t) \mathcal{H}^{(2)}(x)(t) - r(t) f(x(\bar{\varphi}(s))), \quad (3.10)$$

and so any fixed point of  $\mathcal{H}$  is, for large  $t$ , a solution of (1.1). Proceeding by a similar way to that in the proof of Theorem 2.1, we obtain that  $\mathcal{H}(\Omega)$  is relatively compact.

Now we prove the continuity of  $\mathcal{H}$  in  $\Omega$ . Let  $\{x_n\}$ ,  $n \in \mathbb{N}$ , be a sequence in  $\Omega$ , which uniformly converges to  $\bar{x} \in \Omega$  on every compact interval of  $[t_0, \infty)$ . Since

$$\left| \int_\tau^\infty r(s) (f(x_n(\bar{\varphi}(s)))) w(s, \tau) ds \right| \leq KM \int_\tau^\infty |r(s)| ds, \quad (3.11)$$

in virtue of (3.1), the dominated convergence Lebesgue theorem gives

$$\lim_{n \rightarrow \infty} \int_\sigma^\infty \int_\tau^\infty r(s) (f(x_n(\bar{\varphi}(s)))) w(s, \tau) ds d\tau = \int_\sigma^\infty \int_\tau^\infty r(s) (f(x(\bar{\varphi}(s)))) w(s, \tau) ds d\tau. \quad (3.12)$$

Moreover, fixing  $T > t_0$ , we have

$$\begin{aligned}
& \left| \int_t^\infty \int_\sigma^\infty \int_\tau^\infty r(s) (f(x_n(\bar{\varphi}(s))) - f(\bar{x}(\bar{\varphi}(s)))) w(s, \tau) ds d\tau d\sigma \right| \\
& \leq \left| \int_t^T \int_\sigma^\infty \int_\tau^\infty r(s) (f(x_n(\bar{\varphi}(s))) - f(\bar{x}(\bar{\varphi}(s)))) w(s, \tau) ds d\tau d\sigma \right| \\
& \quad + \int_T^\infty \int_\sigma^\infty \int_\tau^\infty |r(s)| (f(x_n(\bar{\varphi}(s))) + f(\bar{x}(\bar{\varphi}(s)))) |w(s, \tau)| ds d\tau d\sigma.
\end{aligned} \tag{3.13}$$

In view of (3.9), we have

$$\int_T^\infty \int_\sigma^\infty \int_\tau^\infty |r(s)| (f(x_n(\bar{\varphi}(s))) + f(\bar{x}(\bar{\varphi}(s)))) |w(s, \tau)| ds d\tau d\sigma \leq 2L_1 \int_T^\infty s |r(s)| ds, \tag{3.14}$$

and so, choosing  $T$  sufficiently large, from (3.13) we obtain the continuity of  $\mathcal{H}$  in  $\Omega$ . Hence, by the Tychonoff fixed point theorem, the operator  $\mathcal{H}$  has a fixed point  $x$ , which is a bounded solution of (1.1) satisfying (3.2).  $\square$

*Remark 3.2.* When  $n = 4$ , Theorem 3.1 extends to equations with deviating argument of a similar result stated in [15] for (1.7). Observe that our approach used here is completely different from that in [15].

The next result shows that, under additional assumptions, condition (3.1) can be also necessary for the existence of bounded AL-solutions of (1.1).

**Theorem 3.3.** *Assume either*

$$r(t) \geq 0, \quad q''(t) \geq 0 \quad \text{for large } t \tag{3.15}$$

or

$$r(t) \leq 0, \quad q''(t) \leq 0 \quad \text{for large } t. \tag{3.16}$$

If

$$\int_0^\infty |r(t)| t dt = \infty, \tag{3.17}$$

then (1.1) does not have solutions  $x$  satisfying

$$0 < \alpha \leq x(t) \leq \beta, \tag{3.18}$$

for large  $t$  and some  $\alpha$  and  $\beta$ . Consequently, every bounded solution  $x$  of (1.1) satisfies

$$\liminf_{t \rightarrow \infty} |x(t)| = 0. \quad (3.19)$$

The following lemmas are needed for proving Theorem 3.3.

**Lemma 3.4.** Assume  $q''(t) \geq 0$  for  $t \geq T \geq 0$ , and let  $x$  be a solution of (1.1) satisfying (3.18) for  $t \geq T$ . Then, there exist two constants  $M_1, M_2$  such that for  $t \geq T$

$$-\int_T^t sq(s)x''(s)ds < tq'(t)x(t) - tq(t)x'(t) + M_1, \quad (3.20)$$

$$\int_T^t (s-T)q'(s)x'(s)ds < M_2. \quad (3.21)$$

If  $q''(t) \leq 0$  for  $t \geq T \geq 0$ , inequalities (3.20), (3.21) hold in the opposite order.

*Proof.* Suppose  $q''(t) \geq 0$  on  $[T, \infty)$ . We have

$$\int_T^t sq(s)x''(s)ds = tq(t)x'(t) - Tq(T)x'(T) - \int_T^t q(s)x'(s)ds - \int_T^t sq'(s)x'(s)ds. \quad (3.22)$$

Since

$$\begin{aligned} \int_T^t sq'(s)x'(s)ds &= tq'(t)x(t) - Tq'(T)x(T) - \int_T^t q'(s)x(s)ds - \int_T^t sq''(s)x(s)ds, \\ \int_T^t q(s)x'(s)ds &= q(t)x(t) - q(T)x(T) - \int_T^t q'(s)x(s)ds, \end{aligned} \quad (3.23)$$

from (3.22), we get

$$\begin{aligned} -\int_T^t sq(s)x''(s)ds &= tq'(t)x(t) - tq(t)x'(t) + q(t)x(t) \\ &\quad - 2 \int_T^t q'(s)x(s)ds - \int_T^t sq''(s)x(s)ds + K_1, \end{aligned} \quad (3.24)$$

where  $K_1$  is a suitable constant. Since  $q, x$  are bounded,  $q''(t) \geq 0$ , in view of (1.1), inequality (3.20) follows.

Moreover,  $q'$  is nondecreasing for  $t \geq T$ . Because  $q$  is a positive bounded function, then  $q'(t) \leq 0$  on  $[T, \infty)$ . Thus, inequality (3.21) follows integrating by parts and using (1.1). Finally, if  $q''(t) \leq 0$  on  $[T, \infty)$ , the argument is similar.  $\square$

**Lemma 3.5.** Let  $x$  be a solution of (1.1) satisfying (3.18) for large  $t$ . If

$$\int_0^\infty |r(t)|dt < \infty, \quad (3.25)$$

then  $x''$  is bounded. If, in addition,  $r(t) \geq 0$ ,  $q''(t) \geq 0$  for  $t \geq T \geq 0$  and (3.17) holds, then for large  $t$

$$x'''(t) + q(t)x'(t) < q'(t)x(t). \quad (3.26)$$

If  $r(t) \leq 0$ ,  $q''(t) \leq 0$  for  $t \geq T \geq 0$ , inequality (3.26) holds in the opposite order.

*Proof.* Since  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ , there exists  $\tau$  such that for  $t \geq \tau$

$$0 < \alpha \leq x(\varphi(t)) \leq \beta. \quad (3.27)$$

Without loss of generality, let  $\tau = T$ . Thus,  $\inf_{t \geq T} f(x(\varphi(t))) > 0$ .

Let  $u$  and  $v$  be two linearly independent solutions of (1.3) with Wronskian  $d = 1$ . By assumptions on  $q$ , all solutions of (1.3) and their derivatives are bounded. Thus, by the variation constant formula, there exist constants  $c_1$  and  $c_2$  such that

$$x''(t) = c_1 u(t) + c_2 v(t) - \int_T^t (u(s)v(t) - u(t)v(s))r(s)f(x(\varphi(s)))ds, \quad (3.28)$$

and, in view of (3.25),  $x''$  is bounded.

Let us prove (3.26), and suppose  $r(t) \geq 0$ ,  $q''(t) \geq 0$  on  $[T, \infty)$ . Multiplying (1.1) by  $t$  and integrating from  $T$  to  $t$ , we get

$$tx'''(t) - x''(t) + \int_T^t sq(s)x''(s)ds = Tx'''(T) - x''(T) - \int_T^t sr(s)f(x(\varphi(s)))ds, \quad (3.29)$$

or, in view of Lemma 3.4,

$$tx'''(t) \leq x''(t) + tq'(t)x(t) - tq(t)x'(t) - \int_T^t sr(s)f(x(\varphi(s)))ds + K_2, \quad (3.30)$$

where  $K_2$  is a suitable constant. Since  $x''$  is bounded and

$$\int_T^t sr(s)f(x(\varphi(s)))ds \geq \inf_{t \geq T} f(x(\varphi(t))) \int_T^t sr(s)ds, \quad (3.31)$$

from (3.17) and (3.30), we have

$$\lim_{t \rightarrow \infty} t(x'''(t) - q'(t)x(t) + q(t)x'(t)) = -\infty, \quad (3.32)$$

which gives the assertion. The case  $r(t) \leq 0$ ,  $q''(t) \leq 0$  on  $[T, \infty)$  can be treated in a similar way.  $\square$

*Proof of Theorem 3.3.* Suppose  $r(t) \geq 0$ ,  $q''(t) \geq 0$  for  $t \geq T \geq 0$ . Without loss of generality, assume also that (3.27) holds for  $t \geq T$ . Define

$$v(t) = x''(t) + q(t)x(t), \quad (3.33)$$

$$z(t) = x'''(t) + q(t)x'(t) - \int_T^t q'(s)x'(s)ds. \quad (3.34)$$

Then,  $z'(t) = -r(t)f(x(\varphi(t))) \leq 0$  and

$$z(t) = z(T) - \int_T^t r(s)f(x(\varphi(s)))ds. \quad (3.35)$$

Since  $q'(t) \leq 0$  for  $t \geq T$ , we have

$$v'(t) \leq z(t) + \int_T^t q'(s)x'(s)ds = z(T) - \int_T^t r(s)f(x(\varphi(s)))ds + \int_T^t q'(s)x'(s)ds. \quad (3.36)$$

*Case I.* Assume

$$\int_0^\infty r(t)dt = \infty. \quad (3.37)$$

Since for  $t \geq T$  we have  $q''(t) \geq 0$  and, as claimed,  $q'(t) \leq 0$ , we get

$$\int_T^t q'(s)x'(s)ds = q'(t)x(t) - q'(T)x(T) - \int_T^t q''(s)x(s)ds \leq -q'(T)x(T). \quad (3.38)$$

Thus, from (3.36), we obtain  $\lim_{t \rightarrow \infty} v'(t) = -\infty$ , that is,  $v$  is unbounded. Hence, in view of (3.33), we obtain a contradiction with the boundedness of  $x$ .

*Case II.* Now assume (3.17) and (3.25). In view of Lemma 3.5, without loss of generality, we can suppose that (3.26) holds for  $t \geq T$ . Then,

$$z(T) = x'''(T) + q(T)x'(T) < q'(T)x(T). \quad (3.39)$$

Hence,  $z(T) < 0$ . Integrating (3.36), we get

$$v(t) \leq v(T) + z(T)(t - T) - \int_T^t (s - T)r(s)f(x(\varphi(s)))ds + \int_T^t (s - T)q'(s)x'(s)ds, \quad (3.40)$$

and, in view of Lemma 3.4, we have

$$v(t) \leq v(T) + z(T)(t - T) + M_2. \quad (3.41)$$

Thus,  $\lim_{t \rightarrow \infty} v(t) = -\infty$ , that is, as before, a contradiction. Finally, the case  $r(t) \leq 0$ ,  $q''(t) \leq 0$  for large  $t$  follows in a similar way.  $\square$

## 4. Applications

Here we present some applications of our results to a particular case of (1.1), namely, the equation

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(\varphi(t))|^\lambda \operatorname{sgn} x(\varphi(t)) = 0 \quad (\lambda > 0), \quad (4.1)$$

jointly with some suggestions for future research.

### 4.1. Coexistence of Both Types of AL-Solutions

Applying Theorems 2.1–3.3 to this equation, we obtain the following.

**Corollary 4.1.** (a) Let  $r(t) \neq 0$  for large  $t$ . Equation (4.1) has unbounded AL-solutions if and only if

$$\int_0^\infty |r(t)|\varphi^\lambda(t)dt < \infty. \quad (4.2)$$

(b) Assume either (3.15) or (3.16). Equation (4.1) has bounded AL-solutions if and only if (3.1) holds.

Corollary 4.1 shows also that the deviating argument can produce a different situation concerning the unboundedness of solutions with respect to the corresponding equation without delay, as the following example illustrates.

*Example 4.2.* In view of Corollary 4.1(a), the equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^2} \left| x(\sqrt{t}) \right|^{3/2} \operatorname{sgn} x(\sqrt{t}) = 0, \quad (4.3)$$

where  $q$  satisfies (i), has unbounded AL-solutions, while the corresponding ordinary equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^2} |x(t)|^{3/2} \operatorname{sgn} x(t) = 0, \quad (4.4)$$

in view of Theorem 2.4, does not have unbounded AL-solutions. Moreover, if in addition  $q''(t) > 0$  for large  $t$ , then from Corollary 4.1(b) (4.3) does not have bounded AL-solutions.

The following example shows that the opposite situation to the one described in Example 4.2 can occur.

*Example 4.3.* Consider the equation

$$x^{(4)}(t) + q(t)x^{(2)}(t) + \frac{1}{(t+1)^3}x(t^2) = 0, \quad (4.5)$$

where  $q$  satisfies (i). From Theorem 3.1, (4.5) has bounded AL-solutions and the same occurs for the corresponding ordinary equation. Nevertheless, in view of Corollary 4.1(a), (4.5) has no unbounded AL-solutions.

Examples 4.2 and 4.3 illustrate also that the coexistence of both AL-solutions for (4.1) can fail. Sufficient conditions for the coexistence of these solutions immediately follow from Corollary 4.1.

**Corollary 4.4.** *Let  $r(t) \neq 0$  for large  $t$ .*

(a) *Assume for large  $t$*

$$\varphi(t) \geq t^{1/\lambda}. \quad (4.6)$$

*If (4.1) has unbounded AL-solutions, then (4.1) also has AS bounded solutions.*

(b) *Assume for large  $t$*

$$\varphi(t) \leq t^{1/\lambda}, \quad \operatorname{sgn} r(t) = \operatorname{sgn} q''(t). \quad (4.7)$$

*If (4.1) has bounded AL-solutions, then (4.1) also has unbounded AL-solutions.*

*For the equation without deviating argument*

$$x^{(4)}(t) + q(t)x''(t) + r(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0 \quad (\lambda > 0), \quad (4.8)$$

*from Corollary 4.4 we get the following.*

**Corollary 4.5.** *Let  $r(t) \neq 0$  for large  $t$ .*

(a) *Assume  $\lambda \geq 1$ . If (4.8) has unbounded AL-solutions, then (4.8) has also bounded AL-solutions.*

(b) *Assume  $0 < \lambda \leq 1$  and  $\operatorname{sgn} r(t) = \operatorname{sgn} q''(t)$  for large  $t$ . If (4.8) has bounded AL-solutions, then (4.8) has also unbounded AL-solutions.*

## 4.2. Comparison with Some Results in [15]

As claimed, the existence of unbounded AL-solutions for (4.8) follows also from Theorem 1 in [15]. For  $n = 4$  this result reads as follows.

**Theorem A.** *If*

$$\int_0^\infty |r(t)|t^{\lambda+1}dt < \infty, \quad (4.9)$$



then there exists a solution  $x$  of (4.8) such that

$$x^{(i)}(t) = t^{(i)} + \varepsilon_i(t), \quad i = 0, \dots, 3, \quad (4.10)$$

where  $\varepsilon_i$  are functions of bounded variation for large  $t$  and  $\lim_{t \rightarrow \infty} \varepsilon_i(t) = 0$ .

Therefore, when  $\varphi(t) \equiv t$ , Theorem 2.1 ensures the existence of unbounded AL-solutions of (4.8) under a weaker condition than (4.9), namely,

$$\int_0^\infty |r(t)| t^\lambda dt < \infty. \quad (4.11)$$

On the other hand, Theorem A gives an asymptotic formula for such solutions.

### 4.3. An Open Problem

Equation (1.1) can admit also other types of nonoscillatory solutions, as the following examples show.

*Example 4.6.* Consider the equation

$$x^{(4)}(t) + x^{(2)}(t) - \frac{2t^2 + 4t + 26}{(t+1)^{7/2}} |x(t)|^{3/2} \operatorname{sgn} x(t) = 0. \quad (4.12)$$

In virtue of Corollary 4.1(b), (4.12) has no bounded AL-solutions. Nevertheless, this equation admits nonoscillatory bounded solutions because  $x(t) = (1+t)^{-1}$  is a solution of (4.12).

*Example 4.7.* Consider the equation

$$x^{(4)}(t) + x^{(2)}(t) + \frac{t^2 + 4t + 10}{(t+2)^4 (\log(t+2))^3} x^3(t) = 0. \quad (4.13)$$

Thus, (3.1) holds, while  $\int_0^\infty t^3 r(t) dt = \infty$ . Hence, in virtue of Corollary 4.1, (4.13) has bounded AL-solutions, but no unbounded AL-solutions. Nevertheless, this equation admits nonoscillatory unbounded solutions because  $x(t) = \log(t+2)$  is a solution of (4.13).

The existence of nonoscillatory solutions  $x$  satisfying either  $\lim_{t \rightarrow \infty} x(t) = 0$  or  $\lim_{t \rightarrow \infty} |x(t)| = \infty$ ,  $\lim_{t \rightarrow \infty} x'(t) = 0$  will be a subject of our next research.

### Acknowledgments

The first and third authors are supported by Grant GAP 201/11/0768 of the Czech Grant Agency. The fourth author is supported by the research Project PRIN09-Area 01, Ordinary Differential Equations, of the Italian Ministry of Education.

## References

- [1] M. Marini, "Criteri di limitatezza per le soluzioni dell'equazione lineare del secondo ordine," *Bollettino della Unione Matematica Italiana*, vol. 11, no. 1, pp. 154–165, 1975.
- [2] J. R. Graef and J. Henderson, "Double solutions of boundary value problems for  $2m^{\text{th}}$ -order differential equations and difference equations," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 873–885, 2003.
- [3] J. R. Graef, C. Qian, and B. Yang, "A three point boundary value problem for nonlinear fourth order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 1, pp. 217–233, 2003.
- [4] K. Kamo and H. Usami, "Nonlinear oscillations of fourth order quasilinear ordinary differential equations," *Acta Mathematica Hungarica*, vol. 132, no. 3, pp. 207–222, 2011.
- [5] T. Kusano and M. Švec, "On unbounded positive solutions of nonlinear differential equations with oscillating coefficients," *Czechoslovak Mathematical Journal*, vol. 39, no. 1, pp. 133–141, 1989.
- [6] K. Takaši and T. Tanigawa, "On the structure of positive solutions of a class of fourth order nonlinear differential equations," *Annali di Matematica Pura ed Applicata*, vol. 185, no. 4, pp. 521–536, 2006.
- [7] W. T. Li and X. Yang, "Classifications and existence criteria for positive solutions of systems of nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 2, pp. 446–462, 2004.
- [8] M. Marini, S. Matucci, and P. Řehák, "On decay of coupled nonlinear differential systems," *Advances in Mathematical Sciences and Applications*, vol. 12, no. 2, pp. 521–533, 2002.
- [9] M. Naito and F. Wu, "On the existence of eventually positive solutions of fourth-order quasilinear differential equations," *Nonlinear Analysis*, vol. 57, no. 2, pp. 253–263, 2004.
- [10] M. Naito and F. Wu, "A note on the existence and asymptotic behavior of nonoscillatory solutions of fourth order quasilinear differential equations," *Acta Mathematica Hungarica*, vol. 102, no. 3, pp. 177–202, 2004.
- [11] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.
- [12] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, vol. 89 of *Mathematics and its Applications (Soviet Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.
- [13] I. T. Kiguradze, "An oscillation criterion for a class of ordinary differential equations," *Differentsial'nye Uravneniya*, vol. 28, no. 2, pp. 207–219, 1992.
- [14] M. Bartušek, M. Cecchi, Z. Došlá, and M. Marini, "Oscillation for third-order nonlinear differential equations with deviating argument," *Abstract and Applied Analysis*, vol. 2010, Article ID 278962, 19 pages, 2010.
- [15] M. Bartušek, M. Cecchi, Z. Došlá, and M. Marini, "Asymptotics for higher order differential equations with a middle term," *Journal of Mathematical Analysis and Applications*, vol. 388, no. 2, pp. 1130–1140, 2012.
- [16] W. Xiong and G. Yue, "Almost periodic solutions for a class of fourth-order nonlinear differential equations with a deviating argument," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1184–1190, 2010.